Quantum limits to the measurement of spacetime geometry

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Abstract: This letter analyzes the limits that quantum mechanics imposes on the accuracy to which spacetime geometry can be measured. By applying the physics of computation to ensembles of clocks, as in GPS, we present a covariant version of the quantum geometric limit, which states that the total number of ticks of clocks and clicks of detectors that can be contained in a four volume of spacetime of radius r and temporal extent t is less than or equal to $rt/\pi\ell_P t_P$, where ℓ_P, t_P are the Planck length and time. The quantum geometric bound limits the number of events or 'ops' that can take place in a four-volume of spacetime and is consistent with and complementary to the holographic bound which limits the number of bits that can exist within a three-volume of spacetime.

Recent advances in the physics of computation have allowed the derivation of a quantum geometric limit, an intrinsically quantum mechanical limit to the accuracy with which the geometry of spacetime can be measured [1-2]. This letter goes more deeply into the derivation of the quantum geometric limit and supplies an explicitly covariant version of this limit. Along the way the quantum geometric limit will be related to previous limits to measuring spacetime [3-4], physics of computation [5-6], string theory [7], and holography [8-12].

Attempts to derive quantum limits to the accuracy of measuring the geometry of spacetime date back at least to the 1950s [3-4]. In all discussions of the limits to measurement, care must be taken in applying arguments that rely on the Heisenberg uncertainty principle. In particular, the Heisenberg-based 'standard quantum limit' may be both standard and quantum, but it is not a fundamental limit to measurement; it can be surpassed using squeezed states and entanglement [1]. Fortunately, techniques from the physics of computation such as the Margolus-Levitin theorem [5-6] can be used to derive limits to the accuracy with which quantum systems can be used to measure spacetime geometry.

The Margolus-Levitin theorem states that the time Δt it takes a quantum system such as a clock to go from one state to an orthogonal state is greater than or equal to $\pi\hbar/2E$, where E is the expectation value of the energy of the system above the ground state energy. By contrast, the Heisenberg limit states that the amount of time it takes a quantum system to go from one state to an orthogonal state is greater than $\pi\hbar/2\Delta E$, where ΔE is the standard deviation in the system's energy. Because it depends on energy rather than spread in energy, the Margolus-Levitin theorem is particularly useful for applications to gravitational systems, as will now be shown.

1. Limits to measuring spacetime

The first question that will be investigated is that of the minimum distance and time that can be measured. One can increase the precision of clocks used to measure time by increasing their energy: the M-L theorem implies that the minimum 'tick length' of a clock with energy E is $\Delta t = \pi\hbar/2E$. Similarly, the wavelength of the particles used to map out space can be decreased by increasing their energy. There appears to be no fundamental physical limit to increasing the energy of the clocks used to measure time and the particles used to measure space, until one reaches the Planck scale, $t_P = \sqrt{\hbar G/c^5} = 5.391 \times 10^{-44}$ seconds, $\ell_P = ct_P$. At this scale, the Compton wavelength $2\pi\hbar/mc$ of the clocks and particles is on the same order of magnitude as their Schwarzschild radius $2mG/c^2$, and

quantum gravitational effects come into play [4]. Accordingly, the Planck length gives a natural cutoff at small length scales [3-5, 8-9]. However, in the absence of a full theory of quantum gravity, such a limit on measuring small length scales must be regarded as heuristic.

The second question is that of the accuracy to which one can map out the large-scale structure of spacetime. Here, the physics of computation can be used to derive limits that go beyond the standard quantum limit. One way to measure the geometry of spacetime is to fill space with a 'swarm' of clocks, exchanging signals with the other clocks and measuring the signals' times of arrival. In this picture, the clocks could be as large as GPS satellites, or as small as elementary particles.

Let's review the arguments of [1-2] to see how accurately this swarm of clocks can map out a sub-volume of spacetime with radius r over time t. Every tick of a clock or click of a detector is an elementary event in which a system goes from a state to orthogonal state. Accordingly, the the total number of ticks and clicks that can take place within the volume is limited by the Margolus-Levitin theorem: it is less than $2Et/\pi\hbar$, where E is the energy of the clocks within the volume, measured from their ground state.

If the clocks are packed too densely, they will form a black hole and be useless for participating in the measurement of spacetime outside their horizon. To prevent black hole formation, the total energy of clocks within a spacelike region of radius r must be less than $E/2Gc^4$. Taken together, these two limits imply that the total number of elementary events that can occur in the volume of spacetime is no greater than

$$\# \equiv \frac{rt}{\pi l_P t_P}. \tag{1}$$

The Planck-scale limit above arises from applying equation (1) to to the case of a single clock. The quantum geometric limit (1) was derived without any recourse to quantum gravity: the Planck scale makes its appearance simply from combining quantum limits to measurement with the requirement that a region not itself be a black hole. If the region is at or above its critical density, then equation (1) still holds as long as r is interpreted as the radius of the horizon of the region as measured by an external observer.

2. Covariant formulation

2.1 Covariant volumes

Now redo the quantum geometric limit (1) in a covariant way. First of all, let's construct a covariant version of the notion of a four volume with temporal extent t and

spatial radius r. To give a covariant version of the temporal extent of a four volume, look at the maximum length of a time-like geodesic contained within the volume. Radii can then be defined with respect to this geodesic in a covariant fashion as follows.

Consider a segment of a time-like geodesic Γ that is not contained within a closed horizon. Let the segment have proper time length t, and let τ be an affine parameter measuring proper time along the geodesic. This geodesic can be used to define covariant 'spheres,' 'cylinders,' and 'solids of rotation' using Einstein's technique for clock synchronization. For example, the sphere of radius x at time τ is given by the intersection of the forward light cone from the point on the geodesic $\tau - x/2c$ with the backward light cone from the point at $\tau + x/2c$. (Because the geodesic is assumed not to sit within a horizon, we don't have to worry about the forward light cone at $\tau - x/2c$ being 'trapped,' and unable to intersect the backward light cone from a later point [11].)

This construction is clearly covariant, and allows us to define spheres of any diameter at any point along the geodesic. The same construction allows us to define a covariant cylinder of radius r and length t centered on Γ as the collection of spheres of radius $x \leq r$ at times $0 \leq \tau \leq t$. Arbitrary 'solids of rotation' of varying radius can similarly be defined by letting the radius of the spheres vary with τ.

A covariant version of the notion of 'a volume of spacetime of radius r over time t' then becomes 'a volume of spacetime contained within the covariant cylinder of radius r centered on a timelike geodesic of proper length t.' With this interpretation of r and t, the geometric quantum limit (1) becomes completely covariant. The covariant volume considered need not be a cylinder: any solid of rotation will do. At each point in time τ the density of matter within the corresponding sphere the solid of rotation should be less than density of a black hole of the same radius.

2.2 Covariant limit to the number of events

Now let's construct a covariant version of the energy-time limit to the number of ticks and clicks. The key point here is that the number of times a clock ticks or a detector clicks is a scalar quantity, agreed upon by all observers. That is, although they may express the quantum operators corresponding to when a clock ticks or detector clicks in different forms, all observers agree whether or not a particular quantum variable has moved from one state to an orthogonal state within a particular spacetime volume. This agreement arises because the form that one observer assigns to a quantum operator is unitarily related to the form that another observer assigns [13].

For example, in an atomic clock, the thing that 'ticks' is a nuclear spin. In the inertial frame of the clock, take two hyperfine levels of an atom, and represent their Hamiltonian by $\hbar\omega\sigma_z/2$. In an atomic clock, the atom is prepared in the state $|\rightarrow\rangle$, which is the $+1$ eigenstate of σ_x. The average energy in this state is $E = \hbar\omega/2$. This state precesses in the $x-y$ plane, evolving in to the orthogonal state $|\leftarrow\rangle$ after a proper time $\Delta t = \pi/\omega = \pi\hbar/2E$. That is, the precessing spin saturates the Margolus-Levitin bound. The process by which the nuclear spin evolves from one state to an orthogonal state constitutes one 'tick' of the clock. (Conventional atomic clocks consist of many spins ticking at the same rate.)

Now consider the same process in a different coordinate system, e.g. a Lorentz-boosted frame or an accelerated frame. In the new coordinate system the observable that corresponds to the clock variable σ_x takes on a new form $\sigma' = U\sigma_x U^\dagger$, where U is a unitary transformation [13]. The new clock variable need no longer correspond to the nuclear spin alone; for example, in a Lorentz-boosted frame, the nuclear spin becomes mixed with the the atom's momentum via the usual Dirac equation transformation [14]. The coordinate time it takes the new clock variable to move from its initial state to an orthogonal state is typically different; but the proper time is the same. In particular, all observers agree on the number of times the clock ticks along a particular segment of the timelike geodesic along which it travels.

In other words, the number of times a clock ticks as it travels a segment of its path is a scalar quantity. Note that this argument depends only on the fact that the clock variable in one coordinate system is unitarily related to the clock variable expressed in another coordinate system. This unitary equivalence holds for non-inertial clocks as well. Similarly, the operator corresponding to a click of a detector, in one coordinate system is unitarily related to the operator that correponds to that event in another coordinate system. Generically, define an 'event' to occur when a state defined by a given quantum operator moves to an orthogonal or almost orthogonal state. The quantum state on a space-like three surface within the four volume is defined relative to some maximal commuting set of operators (e.g., local quantum fields) on that surface. A localized event occurs when some locally defined state (e.g., the spin state of a proton in an atomic clock) moves to an orthogonal state. The number of events that occurs within a four-volume of spacetime is then a scalar quantity.

Since the number of events is a scalar quantity, the limit to this number should be expressable in a covariant fashion. Above, the upper limit to the number of events that

could occur within a volume of spacetime was given by the Margolus-Levitin theorem as $2Et/\pi\hbar$. This limit is not covariant. Note, however, that in the rest frame of an inertial clock, the number of ticks that the clock can make over proper time t is bounded by

$$\# = 2Et/\pi\hbar = (-2/\pi\hbar)\int T^a{}_a dV, \tag{2}$$

where T^{ab} is the energy-momentum tensor (assuming a signature for the metric of $(-,+,+,+)$). The right half of equation (2) is now covariant. In general, the total number of ticks or clicks that can occur within a volume of spacetime is less than or equal to $-2/\pi\hbar$ times the integral of the trace of the energy momentum tensor $T^a{}_a$ over the volume. Equation (2) gives a covariant expression of the Margolus-Levitin theorem.

In the Margolus-Levitin theorem, the energy E is defined to be the energy above the ground state. In constructing the covariant version, equation (2), care must be taken to insure that the ground state energy and is 0 or positive (and $T^a{}_a$ is zero or negative). All known energies are non-negative. If, as can occur in quantum field theory on curved spacetime [13], the ground state energy is negative, a 'cosmological constant' term Λg_{ab} must be added to the energy momentum tensor in equations (1) and (2) to insure that the energy that goes into calculating the maximum number of events is non-negative. The T_{00} part of the resulting renormalized energy momentum tensor then gives the energy above the ground state.

3. Curvature

The covariant expression on the limit on the number of elementary events that can take place within a volume can be combined with Einstein's equations $R_{ab} - (1/2)g_{ab}R = 8\pi G T_{ab}$, to yield further insights into the origin of the quantum geometric limit. Since $T^a{}_a = -(1/8\pi G)R$, the upper limit to the number of ticks and clicks becomes

$$\# = \frac{1}{4\pi^2\hbar G}\int R dV. \tag{3}$$

The covariant expression of the requirement that a sub-volume of clocks within a volume of covariant radius r about a geodesic of length t not form a black hole then becomes

$$\int R dV \leq 4\pi r t. \tag{4}$$

This limit is saturated for a Schwarzschild black hole. Equation (4) provides a simple justification for the covariant version of the quantum geometric limit: if too many events

take place within a region of spacetime, the curvature within that region becomes so large that a horizon forms. After a horizon forms, observers within the region can no longer report on their local geometry to observers outside the region.

4. Relation to holography

The quantum geometric limit (1) is consistent with and complementary to the Bekenstein bound [8], the holographic bound, which states that the maximum number of degrees of freedom that can be contained in a spacelike region is bounded by the area of that region divided by the Planck length squared [9-10], and the covariant entropy bound [11], which limits the amount of information on light sheets.

In contrast with these bounds, which limit the number of *bits* that can be contained in a region, the quantum geometric limit bounds the number of operations or *ops* that can be contained in a four volume of spacetime. In a computation, the number of ops performed is always greater than or equal to the number of bits used in the computation (for a bit to be used, it has to participate in at least one op). The same principle holds here. Indeed, the argument that leads to (1) also implies that maximum number of quanta of wavelength $\lambda \leq 2r$ that can be packed into a volume of radius r without turning that volume into a black hole is bounded by $r^2/\pi l_P^2$, consistent with the Bekenstein bound and holography [12,15].

The idea behind the holographic principle is that the information contained within a three-dimensional spacelike volume can be encoded 'holographically' on its two-dimensional boundary. If the density of bits on the boundary is no greater than one per Planck length squared, then the total number of bits within the volume is limited by the area of the boundary divided by the Planck length squared. The quantum geometric limit dovetails nicely with holography. Imagine that the events that take place within a four volume are projected onto a 1+1 dimensional world sheet intersecting with the volume. If the density of events projected onto this world sheet is no greater than one per Planck length times Planck time, then the quantum geometric limit immediately follows.

Just which world sheet should be used for the projection is an interesting question. The method used above suggests constructing such a world sheet from the longest timelike geodesic within the volume by a process related to Einstein clock synchronization. Just as covariant spheres, cylinders, and solids were constructed from this geodesic out of the intersection of expanding and contracting light cones, we can construct world sheets out of radii of these spheres. A suitable world sheet containing the volume is then given by the

cross-section of the minimal covariant cylinder or solid of rotation containing the volume. The number of ops contained in the volume is then less than or equal to the area of this world sheet divided by the π times the Planck length times the Planck time.

Using covariant solids of rotation to define the upper bound in the quantum geometric limit can overestimate this bound for four volumes that are not spherically symmetric. A tighter bound can be constructed for such volumes by using world sheets that are directly associated with the volume itself. A second plausible way to construct a world sheet to insert in the covariant quantum geometric quantum limit (1) is to use an extremal world sheet of maximum area contained in the volume. Such a world sheet is like an extremal 'string' world sheet, and the number of ops is then limited by the action of the string world sheet (up to a factor given by the value of the dilaton field [7]). Both this method and the Einstein method are covariant, as they rely only on world sheet areas to insert in equation (1). The extremal world sheet method can lead to a tighter bound than the covariant solid of rotation method. However, this method does not possess the immediate justification of the formation of a horizon in terms of Schwarzschild black holes. Further research is necessary to construct the tightest bounds given by the covariant quantum geometric limit for nonuniform spacetime volumes.

We can even insert the area of a *spacelike* surface in (1) by projecting the events within the volume onto a spacelike two surface. In the covariant entropy bound [11], the number of bits of information that can be contained on a contracting light sheet emanating from a convex closed surface of area A is no greater than $A/(4\ln 2)l_P^2$. In regions without horizons that contain no trapped surfaces, there are typically two such contracting light sheets, one forward directed and one backward directed [11]. These light sheets bound a four volume containing the original surface. Applying the quantum geometric limit to this four volume limits the number of ops within the surface to fewer than $A/\pi l_P^2$. But as before, the number of bits associated with the volume (i.e., the number of bits that participate in ops contained within the volume) is less than the number of ops. So the quantum geometric limit is also consistent with the covariant entropy bound.

5. Accuracy of measuring space vs. time

The quantum geometric limit implies that there is a tradeoff between the accuracy with which one can measure time, and the accuracy with which one can measure space. Equation (1) implies that the maximum spatial resolution can only be obtained by relaxing the temporal resolution and having each clock tick only once in time T. This lack of temporal

resolution is characteristic of systems, like black holes, that attain the holographic bound [6]. By contrast, if the events are spread out uniformly in space and time, the number of cells within the spatial volume goes as $(R/l_P)^{3/2}$ – less than the holographic bound – and the number of ticks of each clock over time T goes as $(T/t_P)^{1/2}$. This is the accuracy to which ordinary matter such as radiation and massive particles map out spacetime. Because it is at or close to its critical density, our own universe maps out the geometry of spacetime to an accuracy approaching the absolute limit given by (1): there have been no more than $(T/t_P)^2 \approx 10^{123}$ 'ticks and clicks' since the big bang [6].

In a uniform distribution of events within our universe, the average spacing between clock ticks along timelike geodesics is $\sqrt{Tt_p} \approx 10^{-13}$ seconds, corresponding to a spatial resolution of a little more than 10^{-5} meters. Here on Earth, where the energy densities are much higher than average, the accuracy that can be attained is higher. Out in intergalactic space, however, the accuracy to which quanta are currently mapping out spacetime geometry is given by the wavelength of the primordial black body radiation. In other words, in most places in the universe, spacetime geometry is currently determined to an accuracy of about a millimeter.

6. Quantum computation and quantum gravity

The fundamental physics of computation [5-6] was used to derive the limits to measuring spacetime geometry reported here and in [1-2]. The apparently intimate relationship between quantum computation and the measurement of spacetime geometry suggests attempting to construct a theory of quantum computation based on quantum gravity. Such a theory has recently been constructed [16]; the resulting theory validates the quantum geometric limit explored here.

In the computational theory of quantum gravity, events correspond directly to ops in an underlying quantum computation. Each op flips bits, just as a quantum clock flips bits, and by flipping bits contributes to the curvature of the underlying spacetime. The resulting spacetime is discrete and corresponds to a Regge calculus. (More precisely, a quantum computation can be represented as a sum of 'computational histories,' each of which possesses a definite causal structure and a definite action at each logic gate. Each computational history corresponds to a Regge calculus.)

In this 'computational universe,' the integral of curvature in equation (4) above becomes a sum over triangular hinges in the Regge calculus, and the integral of the trace of the energy-momentum tensor becomes the sum of the angles rotated in the individual

quantum logic gates. The limit of (4) on the total curvature contained within a spacetime volume then translates into a restriction on the areas and deficit angles of hinges in the Regge calculus. The integral of curvature over a discretized volume corresponding to a given hinge is just the area of the hinge A times the deficit angle ϵ of the hinge. If we identify rt in (4) with the area of the hinge, the quantum geometric limit then becomes simply the statement that the deficit angle in Regge calculus is no greater than 2π.

Conclusion

The quantum geometric limit states that the number of elementary events such as clock ticks, detector clicks, or bit flips, that can be contained in a four volume of space time of covariant radius r and spatial extent t is limited by $rt/\pi l_P t_P$, or more generally, by $A/\pi l_P t_P$, where A is the area of a world sheet associated with the volume. By bounding the number of ops that can take place within a four volume, this limit is complementary to the holographic limit, which bounds the number of bits associated with a three volume. The quantum geometric limit can be thought of as a bound on the 'time computational complexity' — the number of ops — of a computation proceeding in that volume, while the holographic limit bounds the 'spatial computational complexity' — the number of bits. The quantum geometric limit is consistent with and supportive of an underlying computational theory of quantum gravity.

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